Adaptive and Constrained Algorithms for Inverse Compositional Active Appearance Model Fitting

— CVPR 2008 Paper Supplemental Material —

George Papandreou and Petros Maragos
School of Electrical and Computer Engineering
National Technical University of Athens, Greece
http://cvsp.cs.ntua.gr

1. Including Priors into Flexible Warp-based Inverse Compositional Algorithms

In this note we expand the discussion of Section 4.1 of the main paper on computing the inverse-compositional to additive parameter update $(4+n) \times (4+n)$ Jacobian matrix $J_{\tilde{\mathbf{p}}}$. Full details are given for the particularly interesting case of the thin-plate spline warp [2].

Our starting point is the relationship $\mathbf{W}(\mathbf{x}; \tilde{\mathbf{p}} + J_{\tilde{\mathbf{p}}} d\tilde{\mathbf{p}}) \approx \mathbf{W}\big(\mathbf{W}(\mathbf{x}; -d\tilde{\mathbf{p}}); \tilde{\mathbf{p}}\big)$, which holds for all points $\mathbf{x}$ in the image plane to first order in $d\tilde{\mathbf{p}}$ [1,4]. Differientiation w.r.t. $d\tilde{\mathbf{p}}$ yields

$$\underbrace{\frac{\partial \mathbf{W}}{\partial \tilde{\mathbf{p}}}\Big|_{(\mathbf{x};\tilde{\mathbf{p}})}}_{2\times(4+n)} \underbrace{J_{\tilde{\mathbf{p}}}}_{(4+n)\times(4+n)} \approx -\underbrace{\frac{\partial \mathbf{W}}{\partial \mathbf{x}}\Big|_{(\mathbf{x};\tilde{\mathbf{p}})}}_{2\times 2} \underbrace{\frac{\partial \mathbf{W}}{\partial \tilde{\mathbf{p}}}\Big|_{(\mathbf{x};\tilde{\mathbf{p}}=\mathbf{0})}}_{2\times(4+n)} . \tag{1}$$

Equation (1) gives $2 \times (4+n)$ constraints per image point $\mathbf{x}$.

Since the warp $\mathbf{W}$ is uniquely determined by positions of the shape landmarks, it suffices to apply Eq. (1) L times, once for the spatial position $\mathbf{x}_l$, $l = 1, \dots, L$ of each landmark on the mean shape $\mathbf{s}_0$. Putting together the L resulting terms in a single block matrix equation yields

$$\underbrace{\begin{bmatrix} \frac{\partial \mathbf{W}}{\partial \tilde{\mathbf{p}}}\Big|_{(\mathbf{x}_1;\tilde{\mathbf{p}})} \\ \vdots \\ \frac{\partial \mathbf{W}}{\partial \tilde{\mathbf{p}}}\Big|_{(\mathbf{x}_L;\tilde{\mathbf{p}})} \end{bmatrix}}_{2L\times(4+n)} \underbrace{J_{\tilde{\mathbf{p}}}}_{(4+n)\times(4+n)} \approx -\underbrace{\begin{bmatrix} \frac{\partial \mathbf{W}}{\partial \mathbf{x}}\Big|_{(\mathbf{x}_1;\tilde{\mathbf{p}})} \frac{\partial \mathbf{W}}{\partial \tilde{\mathbf{p}}}\Big|_{(\mathbf{x}_1;\tilde{\mathbf{p}}=\mathbf{0})} \\ \vdots \\ \frac{\partial \mathbf{W}}{\partial \mathbf{x}}\Big|_{(\mathbf{x}_L;\tilde{\mathbf{p}})} \frac{\partial \mathbf{W}}{\partial \tilde{\mathbf{p}}}\Big|_{(\mathbf{x}_L;\tilde{\mathbf{p}}=\mathbf{0})} \end{bmatrix}}_{2L\times(4+n)} . \tag{2}$$

Denoting as $\frac{\partial \mathbf{W}}{\partial \tilde{\mathbf{p}}}\Big|_{(\mathbf{x}_{1:L};\tilde{\mathbf{p}})}$ the $(2L) \times (4+n)$ stacked matrix of derivatives on the left-hand-side and as $\frac{\partial \mathbf{W}}{\partial \mathbf{x}}\Big|_{(x_{1:L};\tilde{\mathbf{p}})} \odot \frac{\partial \mathbf{W}}{\partial \tilde{\mathbf{p}}}\Big|_{(\mathbf{x}_{1:L};\mathbf{0})}$ the stacked block-by-block matrix product on the right-hand-side of the previous equation, we can write it more compactly as

$$\underbrace{\frac{\partial \mathbf{W}}{\partial \tilde{\mathbf{p}}}\Big|_{(\mathbf{x}_{1:L};\tilde{\mathbf{p}})}}_{2L\times(4+n)} \underbrace{J_{\tilde{\mathbf{p}}}}_{(4+n)\times(4+n)} \approx -\underbrace{\frac{\partial \mathbf{W}}{\partial \mathbf{x}}\Big|_{(x_{1:L};\tilde{\mathbf{p}})} \odot \frac{\partial \mathbf{W}}{\partial \tilde{\mathbf{p}}}\Big|_{(\mathbf{x}_{1:L};\mathbf{0})}}_{2L\times(4+n)} . \tag{3}$$

Solving this with the method of least squares yields the Jacobian estimate

$$J_{\tilde{\mathbf{p}}} = -\left(\frac{\partial \mathbf{W}}{\partial \tilde{\mathbf{p}}}\Big|^T_{(\mathbf{x}_{1:L};\tilde{\mathbf{p}})} \frac{\partial \mathbf{W}}{\partial \tilde{\mathbf{p}}}\Big|_{(\mathbf{x}_{1:L};\tilde{\mathbf{p}})} \right)^{-1} \left(\frac{\partial \mathbf{W}}{\partial \mathbf{x}}\Big|_{(x_{1:L};\tilde{\mathbf{p}})} \odot \frac{\partial \mathbf{W}}{\partial \tilde{\mathbf{p}}}\Big|_{(\mathbf{x}_{1:L};\mathbf{0})} \right) , \tag{4}$$

which is Eq. (22) of our main paper.

We move forward and show how the matrices involved in Eq. (4) can be computed. Regarding the $(2L) \times (4+n)$ matrix $\frac{\partial \mathbf{W}}{\partial \tilde{\mathbf{p}}}\Big|_{(\mathbf{x}_{1:L};\tilde{\mathbf{p}})}$, we need compute the $\frac{\partial \mathbf{W}}{\partial \tilde{\mathbf{p}}}\Big|_{(\mathbf{x};\tilde{\mathbf{p}})}$ Jacobian. Applying the chain rule on $\mathbf{W}(\mathbf{x},\tilde{\mathbf{p}}) = \mathbf{S}_t\big(\mathbf{W}(\mathbf{x},\mathbf{p})\big)$ and considering separately the similarity $\mathbf{t}$ and deformation $\mathbf{p}$ parameters gives

$$\frac{\partial \mathbf{W}}{\partial \tilde{\mathbf{p}}}\Big|_{(\mathbf{x};\tilde{\mathbf{p}})} = \left[\frac{\partial \mathbf{S}}{\partial \mathbf{t}}\Big|_{(\mathbf{W}(\mathbf{x},\mathbf{p});\mathbf{t})} \quad \frac{\partial \mathbf{S}}{\partial \mathbf{x}}\Big|_{(\mathbf{W}(\mathbf{x},\mathbf{p});\mathbf{t})} \cdot \frac{\partial \mathbf{W}}{\partial \mathbf{p}}\Big|_{(\mathbf{x};\mathbf{p})} \right] \tag{5}$$

Taking advantage of the fact that we only need to evaluate the quantities above on the landmark positions $\mathbf{x}_l$, it is easy to show (*c.f.* [4, Sec. 4.1.2]) that

$$\frac{\partial \mathbf{W}}{\partial \tilde{\mathbf{p}}}\Big|_{(\mathbf{x}_{1:L};\tilde{\mathbf{p}})} = \Big[\begin{bmatrix} \mathbf{s}_{\mathbf{p}} & \mathbf{s}_{\mathbf{p}}^{+} & \mathbf{1}_x & \mathbf{1}_x^{+} \end{bmatrix} \quad (1+t_1)\begin{bmatrix} \mathbf{s}_1 & \ldots \mathbf{s}_n \end{bmatrix} + t_2 \begin{bmatrix} \mathbf{s}_1^{+} & \ldots \mathbf{s}_n^{+} \end{bmatrix} \Big], \tag{6}$$

where $\mathbf{s}_{\mathbf{p}} = \mathbf{s}_0 + \sum_{i=1}^{n} p_i \mathbf{s}_i$ is the deformed shape, given the parameters $\mathbf{p}$, $\mathbf{s}^{+}$ denotes the shape $\mathbf{s}$ rotated counter-clockwise by 90^o and $\mathbf{1}_x = \begin{bmatrix} 1 & 0 & \cdots & 1 & 0 \end{bmatrix}^T$ is the shape with 1's in the x-coordinate and 0's in the y-coordinate.

Regarding the $2L \times 2$ matrix $\frac{\partial \mathbf{W}}{\partial \mathbf{x}}\Big|_{(x_{1:L};\tilde{\mathbf{p}})}$, we need compute the Jacobian $\frac{\partial \mathbf{W}}{\partial \mathbf{x}}\Big|_{(\mathbf{x};\tilde{\mathbf{p}})}$. Application of the chain rule on $\mathbf{W}(\mathbf{x},\tilde{\mathbf{p}}) = \mathbf{S}_t\big(\mathbf{W}(\mathbf{x},\mathbf{p})\big)$ gives

$$\frac{\partial \mathbf{W}}{\partial \mathbf{x}}\Big|_{(\mathbf{x};\tilde{\mathbf{p}})} = \frac{\partial \mathbf{S}}{\partial \mathbf{x}}\Big|_{(\mathbf{W}(\mathbf{x},\mathbf{t});\mathbf{t})} \frac{\partial \mathbf{W}}{\partial \mathbf{x}}\Big|_{(\mathbf{x};\mathbf{p})} = \begin{bmatrix} 1+t_1 & -t_2 \\ t_2 & 1+t_1 \end{bmatrix} \frac{\partial \mathbf{W}}{\partial \mathbf{x}}\Big|_{(\mathbf{x};\mathbf{p})}. \tag{7}$$

Computation of the deformation field Jacobian $\frac{\partial \mathbf{W}}{\partial \mathbf{x}}\Big|_{(\mathbf{x};\mathbf{p})}$ depends on the warp family under consideration. For the often used *thin-plate spline* warp [2], we can write the warp function $\mathbf{W}(\mathbf{x},\mathbf{p})$ in the form of a generalized linear model (*c.f.* [3, App. F])

$$\mathbf{W}(\mathbf{x},\mathbf{p}) = \underbrace{W(\mathbf{p})}_{2\times(L+3)} \underbrace{k(\mathbf{x})}_{(L+3)\times 1}, \tag{8}$$

where the vector $k(\mathbf{x})$ is given by

$$k(\mathbf{x}) = \begin{bmatrix} U(|\mathbf{x}-\mathbf{x}_1|) & \ldots & U(|\mathbf{x}-\mathbf{x}_L|) & 1 & x & y \end{bmatrix}^T, \tag{9}$$

$U(r) = r^2 \ln r^2$ is the spline kernel, and $W(\mathbf{p})$ is determined by requiring that the warp maps s_0 to $s_{\mathbf{p}}$. The final result is

$$\frac{\partial \mathbf{W}}{\partial \mathbf{x}}\Big|_{(\mathbf{x};\mathbf{p})} = W(\mathbf{p}) \frac{dk(\mathbf{x})}{d\mathbf{x}}\Big|_{\mathbf{x}} = W(\mathbf{p}) \begin{bmatrix} 2(1+\ln r_{\mathbf{x},1}^2)(\mathbf{x}-\mathbf{x}_1)^T \\ \vdots \\ 2(1+\ln r_{\mathbf{x},L}^2)(\mathbf{x}-\mathbf{x}_L)^T \\ 0 \quad\quad 0 \\ 1 \quad\quad 0 \\ 0 \quad\quad 1 \end{bmatrix}, \tag{10}$$

where $r_{\mathbf{x},l} = \|\mathbf{x}-\mathbf{x}_l\|_2$. We need to evaluate $\frac{\partial \mathbf{W}}{\partial \mathbf{x}}\Big|_{(\mathbf{x};\mathbf{p})}$ for each landmark point $\mathbf{x}_l$. Since the term $k(\mathbf{x})$ does not depend on the shape parameter $\mathbf{p}$, $\frac{dk(\mathbf{x})}{d\mathbf{x}}\Big|_{\mathbf{x}}$ can be pre-computed and be subsequently used in every AAM iteration.

The cost of computing the Jacobian matrix $J_{\tilde{\mathbf{p}}}$ can be analyzed as follows: (a) Computing the $(2L) \times (4+n)$ matrix $\frac{\partial \mathbf{W}}{\partial \tilde{\mathbf{p}}}\Big|_{(\mathbf{x}_{1:L};\tilde{\mathbf{p}})}$ is $\mathcal{O}(nL)$. (b) Computing the $2L \times 2$ matrix $\frac{\partial \mathbf{W}}{\partial \mathbf{x}}\Big|_{(x_{1:L};\tilde{\mathbf{p}})}$ is $\mathcal{O}(L)$. (c) Forming the stacked block-by-block matrix product $\frac{\partial \mathbf{W}}{\partial \mathbf{x}}\Big|_{(x_{1:L};\tilde{\mathbf{p}})} \odot \frac{\partial \mathbf{W}}{\partial \tilde{\mathbf{p}}}\Big|_{(\mathbf{x}_{1:L};\mathbf{0})}$ is $\mathcal{O}(nL)$. (d) Forming the $(4+n) \times (4+n)$ least-squares system matrix $\frac{\partial \mathbf{W}}{\partial \tilde{\mathbf{p}}}\Big|_{(\mathbf{x}_{1:L};\tilde{\mathbf{p}})}^T \frac{\partial \mathbf{W}}{\partial \tilde{\mathbf{p}}}\Big|_{(\mathbf{x}_{1:L};\tilde{\mathbf{p}})}$ is $\mathcal{O}\left(n^2 L\right)$. (e) Inverting the same system matrix is $\mathcal{O}\left(n^3\right)$. Overall, the last two operations dominate the cost of the procedure, whose overall complexity is thus $\mathcal{O}\left(n^2 L + n^3\right)$.

References

[1] S. Baker, R. Gross, and I. Matthews. Lucas-Kanade 20 years on: A unifying framework - Part 4. Technical Report CMU-RI-TR-04-14, Robotics Institute, CMU, 2004.

[2] F. L. Bookstein. Principal warps: Thin-plate splines and the decomposition of deformations. *IEEE Tr. on PAMI*, 11(6):567–585, 1989.

[3] T. F. Cootes and C. J. Taylor. Statistical models of appearance for computer vision. `http://www.isbe.man.ac.uk/~bim/Models/app_models.pdf`, 2004.

[4] I. Matthews and S. Baker. Active appearance models revisited. *Int. J. of Comp. Vision*, 60(2):135–164, 2004.